

# STRUCTURE THEOREMS FOR LINEAR AND NON-LINEAR DIFFERENTIAL OPERATORS ADMITTING INVARIANT POLYNOMIAL SUBSPACES

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**ABSTRACT.** In this paper we derive structure theorems that characterize the spaces of linear and non-linear differential operators that preserve finite dimensional subspaces generated by polynomials in one or several variables. By means of the useful concept of deficiency, we can write explicit basis for these spaces of differential operators. In the case of linear operators, these results apply to the theory of quasi-exact solvability in quantum mechanics, specially in the multivariate case where the Lie algebraic approach is harder to apply. In the case of non-linear operators, the structure theorems in this paper can be applied to the method of finding special solutions of non-linear evolution equations by nonlinear separation of variables.

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## 1. INTRODUCTION

It is a fact that the Schrödinger operators whose point spectrum, or at least part of it, can be computed algebraically are often related to differential operators admitting invariant spaces of polynomials. Lie algebras have played a unifying role in this area, because many of these polynomial spaces turn out to be irreducible modules for a faithful representation of a finite-dimensional Lie algebra by means of first-order differential operators. The classical theory of quasi-exactly solvable potentials has thus been built on the assumption that the exactly solvable Schrödinger operator under study should be expressible as a quadratic element in the universal enveloping algebra of a finite-dimensional Lie algebra of first-order differential operators, admitting an explicitly computable invariant subspace of square-integrable functions, or a complete infinite flag thereof [1–4]. Burnside’s Theorem serves as a strong argument in favor the Lie algebraic approach since it implies that any endomorphism of an irreducible module for a Lie algebra can be represented as a polynomial in the generators of the algebra. However, recent results show that the Lie algebraic approach suffers from various limitations that reduce severely its applicability:

- (1) In the case of polynomial subspaces in one variable, the Lie algebraic approach can only be applied to find the differential operators that leave the polynomial space  $\mathcal{P}_n = \langle 1, z, z^2, \dots, z^n \rangle$  invariant, but it cannot characterize the set of differential operators that map  $\mathcal{P}_n$  into  $\mathcal{P}_m \subset \mathcal{P}_n$  with  $m < n$ . This simple problem has motivated the important notion of *deficiency* used throughout this paper, and applications of it can be found in the construction of solvable classical many-body problems by considering the motion of zeros of polynomials whose coefficients evolve in a controlled manner [5].

- (2) Other subspaces generated by polynomials exist which are not isomorphic to  $\mathcal{P}_n$ . The Lie algebraic approach cannot be applied in these cases, as was already noted by Post and Turbiner, who characterized the spaces of linear differential operators which preserve polynomial subspaces in one variable generated by monomials. In their work [6] they solved this problem with no reference to Lie algebras. The case of a general space spanned by polynomials — referred to as the *generalized Bochner problem* in [6] — remains still open. Somewhat surprisingly this direct approach has not been pursued until very recently, where it has been shown that the class of quasi-exactly solvable potentials is larger than the Lie-algebraic class [7], and that even in Lie-algebraic potentials other non- $\mathfrak{sl}_2$  algebraizations exist which allow to obtain more levels from the energy spectrum of the Hamiltonian, [8]. The existence of differential operators that preserve a general polynomial space and cannot be expressed as quadratic combinations of the generators of  $\mathfrak{sl}_2$  is not in contradiction with Burnside's theorem, since a general polynomial space is not the carrier space for an irreducible representation of  $\mathfrak{sl}_2$ .
- (3) In the case of multi-variable polynomial subspaces, the problem of characterizing the set of linear differential operators that leave these spaces invariant becomes untractable in the Lie algebraic approach. The reason is that, contrary to the single variable case where essentially  $\mathfrak{sl}_2$  is the only algebra of first order differential operators with finite dimensional representations, in more variables many more algebras exist. But moreover, the characterization of second order operators as quadratic combinations of the generators of these algebras requires an extensive analysis of the syzygies corresponding to the primitive ideals associated to the irreducible representations [9]. These problems are entirely bypassed in the direct approach, as shown in Section 3 of this paper, where a simple characterization is given for the set of linear differential operators of any given order  $r$  that leave the simplicial module

$$\mathcal{P}_n = \text{span}\{x_1^{i_1} \dots x_N^{i_N} \mid i_1 + \dots + i_N \leq n\}$$

invariant. Our results coincide with the formulas for the special case  $N = 2$  and  $r = 2$  derived in [9] using the Lie algebraic method.

It has now become clear that the connection to Lie algebras is not an essential feature of exact or partial solvability. Our goal in the first two Sections of this paper is to present a direct method to characterize linear differential operators with invariant polynomial subspaces which is simpler and more powerful than the Lie algebraic approach. We restrict in this paper to the simplest case of polynomial subspaces, namely the simplicial modules  $\mathcal{P}_n$ , the case of general polynomial subspaces shall be treated elsewhere. For any number of variables  $N$ , we provide an explicit basis for the space of linear differential operators of any order  $r$  that map  $\mathcal{P}_n$  into  $\mathcal{P}_m \subseteq \mathcal{P}_n$  with  $m \leq n$ . It should be stressed that although these results allow to construct many differential operators with invariant finite dimensional polynomial subspaces, in general it is not known whether a transformation exists that puts the operator in Schrödinger form. Therefore the results in Sections 2 and 3 are only a first necessary step in the theory of higher dimensional quasi-exact solvability. A general theory would need to face the difficulties of the equivalence problem [10]. Despite this fact, it is worth mentioning that a few examples of partially solvable

multi-dimensional Hamiltonians exist [11–13], which are mostly extensions of the Calogero-Sutherland class.

This paper also addresses the study of nonlinear differential operators with polynomial nonlinearities which possess invariant polynomial subspaces. The motivation for this study is twofold:

- (1) In [14], the important concept of *operator duals* is introduced. Given a finite dimensional space of functions  $\mathcal{F}_n = \text{span}\{f_1, \dots, f_n\}$  which are required to satisfy certain regularity conditions, the operator duals are linear differential operators defined by the relations

$$D_i[f_j] = \delta_j^i.$$

These operators are used in the reduction of non-linear evolution equations to dynamical systems by a method of non-linear separation of variables. In [14] the existence of these differential operators is proved together with results on the regularity of the coefficients. The proof is constructive and therefore given any space  $\mathcal{F}_n$  whose basis elements satisfy the required regularity conditions, non-linear evolution equations can be written which have solutions in the space  $\mathcal{F}_n$ , i.e. special solutions exist of the form  $u(t, x) = \sum_{i=1}^n c_i(t) f_i(x)$  where the coefficients  $c_i(t)$  satisfy a system of coupled non-linear ODEs. However, the order of the operator duals is precisely the dimension of the space  $\mathcal{F}_n$ . The price to pay in order to have invariant spaces of high dimension is that the order of the resulting evolution equations grows with the dimension of the space. For dimensions  $n$  larger than six this reduces the applicative interest of the resulting equations. The motivation coming from [14] is to construct operators that generalize the operator duals, so that the order of the resulting equation and the dimension of the invariant subspace can be independently chosen. In the case of polynomial subspaces  $\mathcal{P}_n$ , the generalization of the operator duals to arbitrary order  $r \leq n$  are the so called *maximal deficiency operators* introduced in Section 2.

- (2) The second motivation comes also from the results on non-linear separation of variables by King, Galaktionov and Svirshchevskii [15–19]. In these papers special interest is given to translation-invariant evolution equations with quadratic non-linearities which admit solutions via non-linear separation of variables. From a physical context, applications are found in nonlinear diffusion and thin film equations. In this paper we extend these results by providing a comprehensive structure theory for autonomous nonlinear operators that preserve a polynomial space  $\mathcal{P}_n$ .

Our paper is organized as follows. In Section 2, we present our direct approach in the case of linear differential operators in one variable. Besides the order of a differential operator, we introduce two key invariants which can be freely specified, which are the degree and the deficiency of the operator relative to a polynomial space  $\mathcal{P}_n$ . The order, degree and deficiency are shown to specify the operator uniquely up to scaling by a constant. The operators of maximal deficiency generalize the operator duals of [14] to any order lower than  $n$ . We give an explicit basis for the space of operators of given order and deficiency. Section 3 is concerned with linear differential operators in  $N$  variables, where all the results of Section 2 are shown to extend to the case of simplicial modules, that is multivariate polynomials of total

degree bounded by a given integer. Section 4 is concerned with non-linear operators preserving polynomial modules, where we give an explicit decomposition theorem for the most general non-linear operator with polynomial coefficients preserving a simplicial module. Section 5 studies the deficiency concept for non-linear operators with polynomial nonlinearities. Section 6 concentrates on non-linear operators that are translation invariant (autonomous) deriving also structure theorems for this class. On Section 7 the application of these results to non-linear separation of variables is discussed while some explicit formulas are given for quadratically non-linear autonomous operators in Appendix A.

## 2. LINEAR OPERATORS IN ONE VARIABLE

In this section, we consider the class of scalar linear differential operators on the real line, with polynomial coefficients. We are interested in the subclass of operators which have a definite *order*, *degree* and *deficiency*. These quantities are defined as follows. The order of

$$(1) \quad L = \sum_{i=0}^r a_i(x) D^i, \quad D := \frac{d}{dx},$$

is as usual the largest  $r$  for which the coefficient  $a_r(x)$  is not identically zero. We say that  $L$  is of degree  $d \in \mathbb{Z}$  if for all  $j \in \mathbb{N}$ , there exists  $c_j \neq 0 \in \mathbb{R}$  not all zero such that

$$(2) \quad L[x^j] = c_j x^{j+d}.$$

In order to define the deficiency, we fix  $n \in \mathbb{N}$  and consider the vector space

$$(3) \quad \mathcal{P}_n = \text{span}\{1, x, \dots, x^n\}$$

of polynomials in  $x$  of degree less than or equal to  $n$ . We say that  $L$  has *deficiency*  $m \in \mathbb{Z}$  relative to  $\mathcal{P}_n$  if

$$(4) \quad L\mathcal{P}_n \subset \mathcal{P}_{n-m}, \text{ but } L\mathcal{P}_n \not\subset \mathcal{P}_{n-m-1}.$$

Let  $\mathcal{L}_{r,m}^{(n)}$  denote the set of linear differential operators with polynomial coefficients, of order less than or equal to  $r$  and of deficiency greater than or equal to  $m$  relative to  $\mathcal{P}_n$ . Again, we emphasize that the notion of operator deficiency only makes sense relative to a particular  $n$ . Most of our discussion will be carried out with the assumption that the  $n$  in  $\mathcal{P}_n$  has been fixed. As such, we will often omit the  $n$  in our terminology and notation, simply speak of the deficiency of an operator, and write  $\mathcal{L}_{r,m}$  instead of  $\mathcal{L}_{r,m}^{(n)}$ .

**Proposition 1.** *The set  $\mathcal{L}_{r,m}$  is a subspace of the vector space of all linear differential operators, i.e., it is closed under linear combinations.*

**Proof** Given linear operators  $L, L' \in \mathcal{L}_{r,m}$ , the order of any linear combination of  $L$  and  $L'$  is less than or equal to  $r$ . Similarly, the deficiency of a linear combination is greater than or equal to  $m$ .  $\square$

**Proposition 2.** *The deficiency of a non-zero linear operator cannot exceed its order.*

**Proof** Suppose that  $L \in \mathcal{L}_{r,m}$  is a linear operator such that  $m > r$ . The operator  $D^{n-m+1}L$  annihilates  $\mathcal{P}_n$ , but has order less than  $n+1$ . This is impossible.  $\square$

Trivial examples of operators with given order, degree and deficiency are given by the operator  $D^i$ , which has order  $i$ , degree  $-i$  and deficiency  $i$ , and the multiplication operator  $x^j$  has order zero, degree  $j$  and deficiency  $-j$ . These operators do not depend on the degree  $n$  of the polynomial space  $\mathcal{P}_n$ . A more significant example, which depends explicitly on  $n, r, m, d$  with  $0 \leq m \leq r \leq n$  and  $-m \leq d \leq r - m$  is the operator

$$(5) \quad L_{rmd} := x^i (n - j - xD)_k D^j, \quad i = r - m, \quad j = r - d - m, \quad k = d + m$$

where we have introduced the Pochhammer operator

$$(6) \quad (n - xD)_k := (-1)^k (xD - n)(xD - (n - 1)) \cdots (xD - (n - k + 1)).$$

A basic result is the following:

**Proposition 3.** *The operator  $L_{rmd}$  has order  $r$ , degree  $d$  and deficiency  $m$ .*

**Proof** Since the  $k$ -fold composition appearing in the right-hand-side of (6) annihilates the monomials  $x^n, \dots, x^{n-k+1}$ , the operator in (6) has order  $k$ , degree zero and deficiency  $k$ . It follows that the operator  $(n - j - xD)_k D^j$  has order  $r = k + j$ , deficiency  $m = k + j$  and degree  $d = -j$ . By left-multiplying by a monomial of  $x$  we raise the degree and lower the deficiency, so that  $L_{rmd}$  has order  $r$ , deficiency  $m$ , and degree  $d$ .  $\square$

Following Proposition 2, it is helpful to refer to an operator whose deficiency equals its order, as a *maximal deficiency operator*. To this end, we will use the symbol

$$(7) \quad K_{rj} = \frac{1}{(r - j)!} L_{r,r,-j} = \frac{1}{(r - j)!} (n - j - xD)_{r-j} D^j, \quad 0 \leq j \leq r \leq n,$$

to denote the operators of maximal deficiency. The normalization constant of  $\frac{1}{(r-j)!}$  will be useful in later formulas. These operators have a number of interesting properties, and play a key role in our theory.

**Proposition 4.** *Up to multiplication by a non-zero real constant, the operator  $K_{rj}$  is the unique  $r^{\text{th}}$  order maximal deficiency operator with polynomial coefficients having degree  $d = -j$ , where  $0 \leq j \leq r$ .*

**Proof** Let  $L$  be an  $r^{\text{th}}$  order maximal deficiency operator with polynomial coefficients and having degree  $d$ . Since  $L$  has polynomial coefficients,  $d \geq -r$ . As well,  $L$  maps  $x^k$  to a multiple of  $x^{k+d}$ . Hence,

$$L[x^k] = 0, \quad n - r - d < k \leq n.$$

However, a non-zero  $r^{\text{th}}$ -order operator can annihilate at most  $r$  monomials. Thus, we have established that  $0 \leq j \leq r$ , where  $j = -d$ .

The leading order term of both  $L$  and  $K_{rj}$  is a multiple of  $x^{r-j} D^r$ . Hence, there exists an  $a \in \mathbb{R}$  such that the order of  $L - aK_{rj}$  is less than  $r$ . However, the deficiency of  $L - aK_{rj}$  is greater than, or equal to  $r$ . Therefore, by Proposition 2, we have  $L = aK_{rj}$ .  $\square$

**Proposition 5.** *For every  $0 \leq j \leq r \leq n$ , we have*

$$(8) \quad (r - j)! K_{rj} = (n - j - xD)_{r-j} D^j = D^j (n - xD)_{r-j}.$$

**Proof** The operators  $(n - j - xD)_{r-j} D^j$  and  $D^j (n - xD)_{r-j}$  both have order  $r$ , deficiency  $r$  and degree  $-j$ . By Proposition 4 they differ by a scalar multiple. By comparing the coefficients of the leading order, we see that these two operators are actually equal.  $\square$

**Proposition 6.** *For a fixed  $r$ , the operators  $K_{rj}$  are recursively defined by*

$$(9) \quad [D, K_{rj}] = -K_{r,j+1}, \quad K_{rr} = D^r.$$

**Proof** Setting  $k = r - j$ , we have, by Proposition 5,

$$\begin{aligned} [D, K_{rj}] &= \frac{1}{(r-j)!} (D^{j+1}(n-xD)_k - (n-j-xD)_k D^{j+1}) \\ &= \frac{1}{(r-j)!} D^{j+1}((n-xD)_k - (n+1-xD)_k) \\ &= -\frac{k}{(r-j)!} D^{j+1}(n-xD)_{k-1} \\ &= -K_{r,j+1}. \quad \square \end{aligned}$$

We can also expand the maximal deficiency operators into an operator sum.

**Proposition 7.** *We have*

$$(10) \quad K_{rj} = \sum_{k=0}^{r-j} (-1)^k \binom{n-k-j}{n-r} \frac{x^k}{k!} D^{k+j}$$

**Proof** Let  $\hat{K}_{rj}$  denote the right hand side of (10). A direct calculation shows that

$$[D, \hat{K}_{rj}] = -\hat{K}_{r,j+1}, \quad \hat{K}_{rr} = D^r.$$

Hence, by Proposition 6,  $K_{rj}$  and  $\hat{K}_{rj}$  have the same recursive definition. Therefore, the two operators are equal.  $\square$

**Proposition 8.** *We have  $\dim \mathcal{L}_{r,r} = r + 1$ . Indeed, the operators  $K_{rj}$ ,  $j = 0, 1, \dots, r$  form a basis for the vector space  $\mathcal{L}_{r,r}$ .*

**Proof** Let  $L \in \mathcal{L}_{r,r}$  be given. Decomposing  $L$  into operator monomials of homogeneous degree, we have by Proposition 2 and the fact that  $\mathcal{P}_n$  is generated by monomials,

$$L = \sum_{j=0}^r a_j K_{rj}.$$

Since the  $K_{rj}$  are linearly independent, we conclude that  $\dim \mathcal{L}_{r,r} = r + 1$ .  $\square$   
The following corollary follows immediately.

**Proposition 9.** *We have*

$$(11) \quad \mathcal{L}_{r,m} = \mathcal{P}_{r-m} \otimes \mathcal{L}_{r,r}, \quad \text{with} \quad \dim \mathcal{L}_{r,m} = (r+1)(r-m+1).$$

*In particular, the set of operators of order  $r$  or less that map  $\mathcal{P}_n$  to itself is of dimension given by  $(r+1)^2 = \dim \mathfrak{gl}(r+1, \mathbb{R})$ .*

We conclude this section with a simple example:

**Example 1.** A basis for the vector space  $\mathcal{L}_{2,2}$  of operators of order two and deficiency two is given by

$$\begin{aligned} K_{22} &= D^2, \\ K_{21} &= (xD - n + 1)D = xD^2 + (1 - n)D, \\ K_{20} &= (xD - n)(xD - n + 1) = x^2D^2 + 2(1 - n)xD + n(n - 1). \end{aligned}$$

### 3. LINEAR OPERATORS IN SEVERAL VARIABLES

The results of the preceding section extend readily to the case of linear operators in  $N$  variables  $(x_1, \dots, x_N)$ . We consider the vector space

$$(12) \quad \mathcal{P}_n^N = \{x_1^{i_1} \dots x_N^{i_N} \mid i_1 + \dots + i_N \leq n\},$$

of polynomials of degree  $n$  in  $N$  variables, of dimension  $\binom{N+n}{n}$ . We shall use the standard multi-index notation whereby given a multi-index  $I = (i_1 \dots i_N) \in \mathbb{N}^N$ , we let

$$(13) \quad x^I := x_1^{i_1} \dots x_N^{i_N}, \quad D_I := \frac{\partial}{\partial x_1^{i_1}} \dots \frac{\partial}{\partial x_N^{i_N}}.$$

The notions of order, degree and deficiency are defined similarly to the single-variable case. Thus, an operator

$$(14) \quad L = \sum_{|I|=0}^r a_I(x_1, \dots, x_N) D_I,$$

will be of degree  $d \in \mathbb{Z}$  if for almost all  $I \in \mathbb{N}^N$ , there exists  $c_{IJ} \neq 0 \in (\mathbb{R}^N)^2$  such that

$$(15) \quad L[x^I] = \sum_{|J|=|I|+d} c_{IJ} x^J.$$

In order to define the deficiency, we fix again  $n \in \mathbb{N}$ , and say that  $L$  has *deficiency*  $m \in \mathbb{Z}$  relative to  $\mathcal{P}_n^N$  if

$$(16) \quad L\mathcal{P}_n^N \subset \mathcal{P}_{n-m}^N, \text{ but } L\mathcal{P}_n^N \not\subset \mathcal{P}_{n-m-1}^N.$$

As in the single variable case, it is easy to see that the deficiency of an operator cannot exceed its order. For example, the operator  $D_I$  has order  $|I|$ , degree  $-|I|$ , and deficiency  $|I|$ , and the operator  $x^I$  has order zero, degree  $|I|$ , and deficiency  $-|I|$ . Again, a more significant example is obtained by introducing the Euler operator

$$(17) \quad E := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i},$$

the Pochhammer operator

$$(18) \quad (n - E)_k := (-1)^k (E - n)(E - (n - 1)) \dots (E - (n - k + 1)),$$

and considering the operator

$$(19) \quad x^I (n - |J| - E)_k D_J.$$

This operator has order  $|J| + k$ , degree  $|I| - |J|$  and deficiency  $|J| + k - |I|$ . Again, if we let  $\mathcal{L}_{r,m}$  denote the vector space of linear differential operators in  $N$  variables

$(x_1, \dots, x_N)$ , with polynomial coefficients, of order  $r$  and deficiency  $m$ , then we have

$$(20) \quad \dim \mathcal{L}_{r,m} = \binom{N+r-m}{r-m} \binom{N+r}{r},$$

or equivalently

$$(21) \quad \mathcal{L}_{r,m} = \mathcal{P}_{r-m} \otimes \mathcal{L}_{r,r}.$$

This formula is in agreement with the result obtained in [9] for operators in two variables, in which it was proved that

$$(22) \quad \dim \mathcal{L}_{r,0} = \binom{2+r}{r}^2.$$

The proof given in [9] was less direct and required an analysis of the syzygies defined by the primitive ideals associated to the irreducible representations of  $\mathfrak{sl}(3, \mathbb{R})$ .

It is easy to see that in contrast with the single variable case, the order, degree and deficiency are not sufficient to characterize an operator uniquely up to a non-zero factor. We have:

**Proposition 10.** *A basis for the vector space of linear differential operators with polynomial coefficients, of order  $r$ , degree  $d$  and deficiency  $m$ , in  $N$  variables is given by*

$$(23) \quad x^I (n - |J| - E)_k D_J, \quad |I| = r - m, |J| = r - d - m, k = d + m.$$

*This vector space is thus of dimension*

$$(24) \quad \binom{N+r-m-1}{r-m} \binom{N+r-m-d-1}{r-m-d}.$$

#### 4. NON-LINEAR OPERATORS

Our objective in this section is to show that the results of the two preceding sections can be applied to prove a structure theorem for a class of non-linear differential operators admitting invariant polynomial subspaces. We shall see that these results complement the structure theorems for operators preserving simplicial modules which were proved in [14].

In dealing with non-linear operators it is convenient to identify differential operators with functions on jet space. To that end, let

$$\mathcal{J}^r(\mathbb{R}) = \mathbb{R} \times \mathbb{R}^{r+1}$$

denote the bundle of  $r$ -jets of smooth maps from  $\mathbb{R}$  to  $\mathbb{R}$ . The  $r$ -th prolongation of a smooth, real-valued function  $f(x)$  is a section of  $\mathcal{J}^r$ , namely

$$\text{pr}_r(f) = (f, Df, D^2f, \dots, D^r f)$$

Thus, the action of an operator on a function of  $x$  is the same thing as the composition of a function of the jet variables with the prolongation:

$$T[f] = T \circ \text{pr}_r(f).$$

We introduce the standard jet coordinates  $x, u_0 = u, u_1 = u_x, u_2 = u_{xx}, \dots, u_r$  on  $\mathcal{J}^r$  so that

$$D^j[f] = u_j \circ \text{pr}_r(f).$$



Henceforth, we fix  $n$ . By Propositions 8 and 9, all linear operators of order  $r$  are expressed uniquely as polynomial linear combinations of the maximal deficiency operators  $K_{rj}$ . Thus, a linear operator  $K_{rj}$  maps  $\mathcal{P}_n$  to  $\mathcal{P}_{n-r}$ ; a quadratically non-linear  $K_{ri}K_{rj}$  maps  $\mathcal{P}_n$  to  $\mathcal{P}_{2(n-r)}$ , a cubically nonlinear  $K_{ri}K_{rj}K_{rk}$  maps  $\mathcal{P}_n$  to  $\mathcal{P}_{3(n-r)}$ , etc. This implies:

**Proposition 11.** *Every operator ( linear or non-linear ) of order  $r$  can be uniquely expressed as*

$$(25) \quad T := p(x) + \sum_i p_i(x)K_{ri} + \sum_{i \leq j} p_{ij}(x)K_{ri}K_{rj} + \sum_{i \leq j \leq k} p_{ijk}(x)K_{ri}K_{rj}K_{rk} + \cdots,$$

The operator in question will map  $\mathcal{P}_n$  to  $\mathcal{P}_n$  if and only if

$$(26) \quad p \in \mathcal{P}_n, \quad p_i \in \mathcal{P}_r, \quad p_{ij} \in \mathcal{P}_{2r-n}, \quad p_{ijk} \in \mathcal{P}_{3r-2n}, \dots$$

The above proposition has the following obvious consequence:

**Corollary 1.** *An operator of order  $r < \frac{n}{2}$ , mapping  $\mathcal{P}_n$  to  $\mathcal{P}_n$  is necessarily linear. If  $r < \frac{2n}{3}$ , then the operator will have at most quadratic non-linearities.*

Conversely, the above proposition can be used to bound the degree of any polynomial space which can be left invariant by a non-linear operator. We have, for example:

**Corollary 2.** *A second-order operator will preserve a polynomial space of degree at most four.*

The extension of these results to multivariate differential operators acting on simplicial polynomial modules is straightforward. We are interested in writing down all non-linear differential operators of order  $r$  that preserve

$$(27) \quad \mathcal{P}_n = \{x_1^{i_1} \dots x_N^{i_N} \mid i_1 + \dots + i_N \leq n\}.$$

We recall that for fixed order  $r$  the maximum deficiency that an operator can attain is  $r$ , which is achieved by any of the following *maximum deficiency* operators:

$$(28) \quad K_J \in \mathcal{L}_{r,r}, \implies K_J = (n - |J| - E)_{r-|J|} D^J,$$

$$(29) \quad J = \{j_1, \dots, j_N \mid j_1 + \dots + j_N \leq r\}.$$

The extension of Proposition 11 to the case of multivariate differential operators and polynomial modules is straightforward by just substituting the simple indices  $i, j$  into multi-indices  $I, J$ , each of which can assume  $\dim \mathcal{L}_{r,r} = \binom{N+r}{r}$  different values. Similarly, the two Corollaries hold verbatim in the multivariate case.

**Example 2.** Write down all second order operators with quadratic non-linearities that map  $\mathcal{P}_4$  into itself. A direct application of Proposition 11 allows to write:

$$(30) \quad T[u] = \sum_{0 \leq i \leq j \leq 2} p_{ij} K_{2i} K_{2j}$$

where  $p_{ij} = p_{ji}$  are constants, and where

$$K_{20} = 6u_0 - 3xu_1 + \frac{1}{2}x^2u_2,$$

$$K_{21} = 3u_1 - xu_2,$$

$$K_{22} = u_2$$

The resulting quadratic combination is the following six parameter family of operators:

$$\begin{aligned} T = & p_{00}u_2^2 + p_{01}(-xu_2^2 + 3u_1u_2) + p_{02}(x^2u_2^2 - 6xu_1u_2 + 12u_2u_0) \\ & + p_{11}(x^2u_2^2 + 9u_1^2 - 6xu_1u_2) \\ & + p_{12}(-x^3u_2^2 + 9x^2u_1u_2 - 6x(2u_2u_0 + 3u_1^2) + 36u_1u_0) \\ & + p_{22}\left(\frac{1}{4}x^4u_2^2 - 3x^3u_1u_2 + 3x^2(2u_2u_0 + 3u_1^2) - 36xu_1u_0 + 36u_0^2\right) \end{aligned}$$

If we are interested in finding only autonomous non-linear equations we need to consider the translation-invariant subfamily of the above operators, i.e. operators where the variable  $x$  does not appear in the coefficients. Imposing this condition leads to

$$p_{10} = p_{12} = p_{22} = 0; \quad p_{11} + p_{20} = 0,$$

so that the only autonomous second order non-linear operators which preserve the space  $\mathcal{P}_4$  are:

$$(31) \quad \{u_{xx}^2, 4u_{xx}u - 3u_x^2\}.$$

The first of these two operators is easily seen to map  $\mathcal{P}_4$  into itself. The second operator is more interesting. Each of the non-linear terms  $u_x^2$  and  $u_{xx}u$  transform a 4<sup>th</sup>-degree polynomial into a 6<sup>th</sup>-degree polynomial. However, the linear combination  $4u_{xx}u - 3u_x^2$  cancels the coefficients of degree 6 and 5, and hence defines a map from  $\mathcal{P}_4$  to itself (see also [15] and [19]).

We investigate further in the analysis of autonomous non-linear operators with invariant polynomial subspaces in the following section.

## 5. THE ALGEBRA OF POLYNOMIALLY NON-LINEAR OPERATORS

In this section we continue the study non-linear operators that are polynomial in the function and its first  $n$  derivatives. First, let us fix some notation. Let

$$\mathcal{T} = \mathbb{R}[x, u_0, \dots, u_n]$$

denote the commutative algebra of non-linear operators that can be expressed as polynomials over  $\mathbb{R}$  in  $x$  and the derivatives  $u_j$ . Multiplication in this algebra is by pointwise multiplication, rather than operator composition. We grade this algebra by total degree in the  $u_j$  variables:

$$\mathcal{T} = \bigoplus_{\ell=0}^{\infty} \mathcal{T}_{\ell},$$

where

$$\mathcal{T}_{\ell} = \text{span}\{x^j u_{i_1} u_{i_2} \cdots u_{i_{\ell}} \mid 0 \leq j < \infty\}.$$

We will refer to the integer  $\ell$  as an operator's degree of non-linearity. Thus,  $\mathcal{T}_1$  is the vector space of linear operators,  $\mathcal{T}_2$  the vector space of quadratically non-linear operators, etc. The vector space  $\mathcal{T}_0$  is the space of constant operators. For example, the operator  $x$  maps all of  $\mathcal{P}_n$  to  $x$ . Thus,  $\mathcal{T}_0$  is a subalgebra of  $\mathcal{T}$ , which is isomorphic to the polynomial algebra in the variable  $x$ . All the other  $\mathcal{T}_{\ell}$ ,  $\ell \geq 1$  are merely subspaces of  $\mathcal{T}$ , not subalgebras.

We further grade each  $\mathcal{T}_{\ell}$  according to the following monomial weighting scheme:

$$(32) \quad \text{wt}(u_i) = n - i, \quad \text{wt}(x) = 1.$$

Thus,

$$(33) \quad \mathcal{T}_\ell = \bigoplus_{k=0}^{\infty} \mathcal{T}_{\ell,k},$$

where

$$(34) \quad \mathcal{T}_{\ell,k} = \text{span}\{x^j u_{i_1} u_{i_2} \cdots u_{i_\ell} \mid j + n\ell - \sum_{s=1}^{\ell} i_s = k\}$$

is the subspace generated by monomials having weight  $k$ . We will refer to the integer  $n - k$  as the *monomial deficiency*. If  $M = x^j u_{i_1} u_{i_2} \cdots u_{i_\ell}$  is an operator monomial with

$$(35) \quad k = \text{wt}(M) = j + n\ell - \sum_{s=1}^{\ell} i_s,$$

then, in accord with the above-introduced meaning of deficiency,  $M$  maps  $\mathcal{P}_n$  into  $\mathcal{P}_k$ , but not into  $\mathcal{P}_{k-1}$ .

In other words, every operator  $T \in \mathcal{T}$  admits the unique decomposition

$$(36) \quad T = \sum_{\ell,k} T_{\ell,k},$$

where

$$(37) \quad T_{\ell,k} = \sum_{\substack{0 \leq i_1 \leq i_2 \leq \cdots \leq i_\ell \leq n \\ k = j + n\ell - (i_1 + i_2 + \cdots + i_\ell)}} C_{ji_1 i_2 \dots i_\ell} x^j u_{i_1} u_{i_2} \cdots u_{i_\ell},$$

and where the sum is taken over finitely many values of  $\ell$  and  $k$ . For generic values of the coefficients  $C_{ji_1 \dots i_\ell}$ , the operator  $T_{\ell,k}$  has deficiency  $n - k$ . However, for certain very specific values of the coefficients, the actual deficiency is greater than the monomial deficiency. This is so because in the linear combination (37) there might occur some cancellations in the terms of highest degree. We see for instance in Example 2 that relative to  $\mathcal{P}_4$ , the operator

$$4u_{xx}u - 3u_x^2 = 4u_2u_0 - 3u_1^2$$

has monomial deficiency  $4 - k = 4 - 8 + 2 = -2$ . However, the actual deficiency of this operator is zero.

Next, we describe generators for  $\mathcal{T}$  that will allow us to precisely determine the deficiency of an operator. Following Proposition 7, let us re-introduce the linear,  $n^{\text{th}}$ -order operators of maximal deficiency:

$$(38) \quad v_j = K_{nj} = \sum_{i=0}^{n-j} (-1)^i \frac{x^i}{i!} u_{i+j} \quad j = 0, \dots, n.$$

Thus,

$$(39) \quad v_n = u_n,$$

$$(40) \quad \begin{aligned} v_{n-1} &= u_{n-1} - xu_n, \\ v_{n-2} &= u_{n-2} - xu_{n-1} + \frac{1}{2}x^2u_{n-2}, \\ v_{n-3} &= u_{n-3} - xu_{n-2} + \frac{1}{2}x^2u_{n-3} - \frac{1}{6}x^3u_{n-4}, \\ &\vdots \end{aligned}$$

These operators transform all elements of  $\mathcal{P}_n$  into a constant. In other words, the  $v_j$  are the operator duals [14] to the monomial basis of  $\mathcal{P}_n$ :

$$v_j[x^k/k!] = \delta_j^k, \quad k = 0, 1, \dots, n.$$

More, generally let us write

$$(41) \quad \tilde{u}_j(t) = \sum_{i=0}^{n-j} u_{i+j} \frac{t^i}{i!} \quad j = 0, \dots, n.$$

In this way,

$$v_j = \tilde{u}_j(-x).$$

**Proposition 12.** *We have*

$$(42) \quad \tilde{u}_j(s+t) = \sum_{i=0}^{n-j} \tilde{u}_{i+j}(s) \frac{t^i}{i!}.$$

**Proof** We have

$$\begin{aligned} \tilde{u}_j'(t) &= \tilde{u}_{j+1}(t), \quad j = 0, 1, \dots, n-1, \\ \tilde{u}_n'(t) &= 0, \\ \tilde{u}_j(0) &= u_j. \end{aligned}$$

Hence,  $u_j \mapsto \tilde{u}_j(t)$  defines a 1-parameter transformation group of  $\mathcal{T}$ . The desired result follows immediately.  $\square$

Now, we can invert the relations (38), and express the  $u_j$  in terms of the  $v_j$ .

**Proposition 13.** *For  $j = 0, \dots, n$ , we have*

$$(43) \quad u_j = \sum_{i=0}^{n-j} v_{i+j} \frac{x^i}{i!},$$

**Proof** We apply (42) with  $s = -x$  and  $t = x$ .  $\square$

Proposition 13 shows that  $x, v_0, \dots, v_n$  freely generate the algebra  $\mathcal{T}$ . The relations (38) and (43) are homogeneous relative to the weights (32). Hence, setting

$$(44) \quad \text{wt}(v_j) = n - j,$$

we recover the grading by monomial deficiency relative to this basis. We now deepen the grading by defining

$$\mathcal{T}_{\ell,k,m} = \text{span}\{x^m v_{i_1} v_{i_2} \cdots v_{i_\ell} \mid m + n\ell - \sum_{s=1}^{\ell} i_s = k\},$$

so that

$$\mathcal{T}_{\ell,k} = \bigoplus_{m=0}^k \mathcal{T}_{\ell,k,m}.$$

**Proposition 14.** *The elements of  $\mathcal{T}_{\ell,k,m}$  have monomial deficiency  $n-k$  and actual deficiency  $n-m$ . Consequently, every operator  $T \in \mathbb{R}[x, u_0, \dots, u_n]$  has deficiency  $n-m$ , where  $m$  is the  $x$ -degree of the polynomial*

$$T = Q(x, v_0, \dots, v_n)$$

that expresses  $T$  relative to the  $x, v_j$  basis.

**Proof** Since the relations (38)–(43) are homogeneous relative to the weighting scheme (32)–(44), the elements of  $\mathcal{T}_{\ell,k,m}$  have weight  $k$ , and hence have monomial deficiency  $n-k$ . Since the operators  $v_j$  map all polynomial to constants, an operator  $Q(x, v_0, \dots, v_n) \in \mathcal{T}$ , having  $m$  as its  $x$ -degree, maps  $\mathcal{P}_n$  to  $\mathcal{P}_m$ , but not to  $\mathcal{P}_{m-1}$ . Therefore such an operator has deficiency  $n-m$ .  $\square$

The key application of the above grading has to do with the decomposition of an operator according to monomial deficiency. Our analysis would be greatly simplified if we could be certain that the decomposition of a non-linear operator  $T$  according to monomial deficiency respects the actual deficiency. In other words, when considering operators of a fixed deficiency, no generality is lost by considering operators that are homogeneous in degree of non-linearity and monomial deficiency.

**Corollary 3.** *Let  $T$  be a non-linear operator whose deficiency is  $n-m$  or more, i.e.,  $T$  maps  $\mathcal{P}_n$  into  $\mathcal{P}_m$ . Let  $\mathcal{T}_{\ell,k}$  be the summands of the decomposition of  $T$  according to degree of non-linearity and monomial deficiency as per (36)–(37). Then, each  $\mathcal{T}_{\ell,k}$  also maps  $\mathcal{P}_n$  into  $\mathcal{P}_m$ .*

## 6. AUTONOMOUS, NON-LINEAR OPERATORS

Our main focus in this section is the subalgebra

$$(45) \quad \mathcal{A} = \mathbb{R}[u_0, \dots, u_n] \subset \mathcal{T}$$

of translation-invariant non-linear operators. The subalgebra inherits the bi-grading relative to degree of non-linearity and monomial deficiency, with

$$(46) \quad \mathcal{A} = \bigoplus_{\ell=0}^{\infty} \bigoplus_{k=0}^{n\ell} \mathcal{A}_{\ell,k},$$

where

$$(47) \quad \mathcal{A}_{\ell,k} = \mathcal{A} \cap \mathcal{T}_{\ell,k} = \text{span}\{u_{i_1} u_{i_2} \cdots u_{i_\ell} \mid n\ell - \sum_{s=1}^{\ell} i_s = k\}$$

Our key result in this section is the characterization of the deficiency of autonomous operators. In other words, we describe  $\mathcal{A} \cap \mathcal{T}_{\ell,k,m}$ .

The obvious approach to construct non-linear operators of deficiency  $m$  would be to write a generic polynomial  $p(x) \in \mathcal{P}_n$  with indeterminate coefficients, act on it by (36) where the degree of the possible non-linearities is bounded by Proposition 11 and impose that the coefficients of all the terms in  $x^j$  for  $j > n-m$  vanish. However, based on the useful concept of deficiency we choose here to adopt a somewhat different approach.

We seek operators that are both translation-invariant and that have maximum deficiency. To this end, we define

$$(48) \quad \xi = \frac{u_{n-1}}{u_n};$$

$$(49) \quad I_{n-j} = \tilde{u}_j(-\xi) = \sum_{i=0}^{n-j} (-1)^k u_{i+j} \frac{\xi^i}{i!}, \quad j = 0, \dots, n,$$

and note that

$$(50) \quad I_0 = u_n, \quad I_1 = 0.$$

The jet space function  $\xi$  is only defined on open neighborhood  $u_n \neq 0$  of  $\mathcal{J}^n$ . Thus, the operators  $I_j$  are defined only for elements of  $\mathcal{P}_n^\times = \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ , the set of polynomials of degree exactly equal to  $n$ . We can still speak of the deficiency of such operator, but this has to be understood in terms of  $\mathcal{P}_n^\times$  rather than  $\mathcal{P}_n$ .

**Proposition 15.** *The non-linear operators  $I_2, I_3, \dots, I_n$  are translation-invariant, and have maximum deficiency  $n$ . In other words, these autonomous, nonlinear operators transform every  $n^{\text{th}}$  degree polynomial into a constant.*

**Proof** By definition,

$$v_j = \tilde{u}_j(-x), \quad I_{n-j} = \tilde{u}_j(-\xi).$$

As well,

$$(51) \quad \xi = x + \frac{v_{n-1}}{v_n}.$$

Hence, Proposition 12 implies that in addition to (49) we also have

$$(52) \quad I_{n-j} = \sum_{k=0}^{n-j} (-1)^k \frac{1}{k!} v_{k+j} \left( \frac{v_{n-1}}{v_n} \right)^k.$$

Hence, the operators  $I_j$  are polynomials of the  $v_j$  divided by a certain power of  $v_n = u_n$ . Therefore, these operators are both translation invariant and of maximal deficiency.  $\square$

We can also invert the relations (49), and express the  $u_j$  in terms of the  $I_j$ .

**Proposition 16.** *For  $j = 0, \dots, n$ , we have*

$$(53) \quad u_j = \sum_{i=0}^{n-j} I_{n-i-j} \frac{\xi^i}{i!},$$

$$(54) \quad = \sum_{i=0}^{n-j} I_i \frac{\xi^{n-j-i}}{(n-j-i)!},$$

**Proof** We apply (42) with  $s = -\xi$  and  $t = \xi$ .  $\square$

Thus, relations (49) (53) tell us that the operators  $I_0, \xi, I_2, \dots, I_n$  also generate the algebra of autonomous operators. These relations are homogeneous with respect to the monomial weights defined in (32). Hence, setting

$$(55) \quad \text{wt}(I_j) = j, \quad \text{wt}(\xi) = 1$$

we recover the grading by monomial deficiency relative to this basis.

Unfortunately, the operators  $I_j, \xi$  are not polynomials in the  $u_j$ , and hence are *not* elements of  $\mathcal{A}$ . Let us therefore consider the larger algebra

$$(56) \quad \hat{\mathcal{A}} = \mathbb{R}[I_0, \xi, I_2, I_3, \dots, I_n]$$

generated by  $\xi$  and the autonomous operators of maximal deficiency. Thanks to (53) we know that  $\mathcal{A} \subset \hat{\mathcal{A}}$ , but this inclusion is strict.

We now deepen the grading of  $\hat{\mathcal{A}}$  by defining

$$\hat{\mathcal{A}}_{\ell,k,m} = \text{span}\{\xi^m I_{i_1} \cdots I_{i_\ell} \mid m + i_1 + \cdots + i_\ell = k\},$$

so that

$$\hat{\mathcal{A}} = \bigoplus_{\ell=0}^{\infty} \bigoplus_{k=0}^{n\ell} \bigoplus_{m=0}^k \hat{\mathcal{A}}_{\ell,k,m}.$$

**Proposition 17.** *We have  $\hat{\mathcal{A}}_{\ell,k,m} = \hat{\mathcal{A}} \cap \mathcal{T}_{\ell,k,m}$ , i.e., the elements of  $\hat{\mathcal{A}}_{\ell,k,m}$  have monomial deficiency  $n - k$  and actual deficiency  $n - m$ .*

**Proof** This follows from (51)–(52), and the fact that the operators  $v_j$  have maximal deficiency.  $\square$

In other words, the deficiency of an autonomous operator is  $n$  minus the  $\xi$ -degree of the polynomial that expresses that operator relative to the  $\xi, I_j$  basis. We are left with the question of the nature of the inclusion of  $\mathcal{A}$  in  $\hat{\mathcal{A}}$ . In other words, which polynomials in  $\xi, I_j$  define true polynomial operators.

**Theorem 1.** *Let  $T = P(x, u_0, \dots, u_n) \in \mathcal{T}$  be a non-linear operator, and let  $T = Q(x, v_0, \dots, v_n)$  be the expression of this operator relative to the non-autonomous generators  $x, v_j$ . Then  $T$  is autonomous, i.e.  $P$  is independent of the variable  $x$  if and only if*

$$T = Q(\xi, I_n, \dots, I_2, 0, I_0).$$

*In this case, the deficiency of  $T$  is equal to  $n$  minus the  $\xi$ -degree of the polynomial  $Q(\xi, I_n, \dots, I_0)$ .*

**Proof** By Propositions 13 and 16, we have

$$u_j = \sum_{i=0}^{n-j} v_{i+j} \frac{x^i}{i!} = \sum_{i=0}^{n-j} I_{n-i-j} \frac{\xi^i}{i!}, \quad j = 0, 1, \dots, n,$$

where, as we noted before,  $I_1 = 0$ , and  $I_0 = v_n = u_n$ .  $\square$

**Example 3.** Let us recast the analysis began in Example 2 in terms of the above operator bases. Relations (43) and (53) take the form

$$\begin{aligned} u_4 &= v_4 & &= I_0; \\ u_3 &= v_3 + xv_4 & &= \xi I_0, \\ u_2 &= v_2 + xv_3 + \frac{1}{2}x^2v_4 & &= I_2 + \frac{1}{2}\xi^2 I_0 \\ u_1 &= v_1 + xv_2 + \frac{1}{2}x^2v_3 + \frac{1}{6}x^3v_4 & &= I_3 + \xi I_2 + \frac{1}{6}\xi^3 I_0; \\ u_0 &= v_0 + xv_1 + \frac{1}{2}x^2v_2 + \frac{1}{6}x^3v_3 + \frac{1}{24}x^4v_4 & &= I_4 + \xi I_3 + \frac{1}{2}\xi^2 I_2 + \frac{1}{24}\xi^4 I_0. \end{aligned}$$

Our goal is to write down all autonomous operators with quadratic non-linearity and zero deficiency, i.e., operators that map  $\mathcal{P}_4$  into itself. By Proposition 17 and the above relations, we are obliged to consider polynomials that are quadratic in  $I_0, I_1, I_2, I_4$  and that have degree 4 or less in the  $\xi$  variable. The question is: which operators of such form are quadratic in  $u_0, u_1, u_2, u_3, u_4$ ? By Corollary

3, no generality is lost by considering operators of a fixed monomial deficiency. Evidently, if the monomial deficiency is 0 or more, the operator will preserve  $\mathcal{P}_4$ . Hence, we must consider operators having monomial deficiency  $-4, -3, -2, -1$ , which corresponds to  $k = 8, 7, 6, 5$ , respectively. The most general operator having  $k = 8$  is a multiple of the monomial  $(u_0)^2$ . Such an operator will have actual deficiency of  $-4$  as well. Similarly reasoning holds for operators with  $k = 7$ ; these are multiples of  $u_0 u_1$ . Let us consider the case  $k = 6$ . The ansatz is now

$$\begin{aligned} C_{02} u_0 u_2 + C_{11} u_1^2 &= \frac{1}{144} (3C_{02} + 4C_{11}) v_4^2 x^6 + \frac{1}{24} (3C_{02} + 4C_{11}) v_3 v_4 x^5 + \text{l.o.t.} \\ &= \frac{1}{144} (3C_{02} + 4C_{11}) I_0^2 \xi^6 + 0 \xi^5 + \text{l.o.t.} \end{aligned}$$

Hence, such an operator preserves  $\mathcal{P}_4$  if and only if is a multiple of  $4u_0 u_2 - 3u_1^2$ , as has already been proven by other methods.

Finally, let us consider the case  $k = 5$ . Now, the ansatz is

$$\begin{aligned} C_{03} u_0 u_3 + C_{12} u_1 u_2 &= \frac{1}{24} (C_{03} + 2C_{12}) v_4^2 x^5 + \frac{5}{24} (C_{03} + 2C_{12}) v_3 v_4 x^4 + \text{l.o.t.} \\ &= \frac{1}{24} (C_{03} + 2C_{12}) I_0^2 \xi^5 + 0 \xi^4 + \text{l.o.t.} \end{aligned}$$

Hence, such an operator preserves  $\mathcal{P}_4$  if and only if is a multiple of  $2u_0 u_3 - u_1 u_2$ . Indeed, because  $I_1 = 0$ , the above calculation shows that this operator has deficiency 1, i.e. it maps  $\mathcal{P}_4$  into  $\mathcal{P}_3$ .

**Autonomous operators with quadratic non-linearities.** We restrict from here on to  $\mathcal{T}_2$ , the vector space of operators with homogeneously quadratic non-linearity. We present a complete characterization of such operators in terms of deficiency, thereby extending the results of Svirshchevskii [19] and Galaktionov [15]. Our analysis can be extended to operators with higher non-linearities, but this shall be treated elsewhere.

Following (33), we let

$$\mathcal{Q} = \bigoplus_{k=-n}^n \mathcal{Q}_k$$

be the linear space of quadratic autonomous non-linear operators up to order  $n$ , graded according to monomial deficiency. Thus,

$$(57) \quad \mathcal{Q} = \mathcal{A}_2 = \text{span}\{u_i u_j \mid 0 \leq i \leq j \leq n\},$$

$$(58) \quad \dim \mathcal{Q} = \binom{n+2}{2},$$

$$(59) \quad \mathcal{Q}_k = \mathcal{A}_{2, n-k} = \text{span}\{u_i u_j \mid i + j = n + k, \quad 0 \leq i, j \leq n\},$$

$$(60) \quad \dim \mathcal{Q}_k = \left\lfloor \frac{n - |k|}{2} \right\rfloor + 1,$$

where and  $\lfloor \cdot \rfloor$  denotes the floor function.

Each of the elements of the above basis of  $\mathcal{Q}_k$  has a different order  $r$ , the minimum and maximum orders for each  $k$  being:

$$(61) \quad r_{\min}^{(k)} = \left\lfloor \frac{n+k}{2} \right\rfloor, \quad r_{\max}^{(k)} = \min(n, n+k)$$



All of the above basis elements have the same deficiency  $k$ , i.e. they map  $\mathcal{P}_n$  into  $\mathcal{P}_{n-k}$ . However, Example 2 and Example 3 show that within each  $\mathcal{Q}_k$  there can be elements whose deficiency is greater than  $k$ .

Remarkably, each  $\mathcal{Q}_k$  possesses an adapted basis that further grades it according to operator order and deficiency. In this regard, for  $r_{\min}^{(k)} \leq r \leq r_{\max}^{(k)}$ , let us introduce the operators

$$(62) \quad Q_{k,r} = \sum_{i=k+n-r}^r (-1)^i \binom{i-k+1}{n-r+1} \binom{n-i}{n-r} u_i u_{n+k-i},$$

We are now ready to state the main result of this Section:

**Theorem 2.** *The operator  $Q_{k,r}$  has monomial deficiency  $k$ , order  $r$ , and deficiency*

$$(63) \quad m(k, r) = k + 2 \left( r - r_{\min}^{(k)} \right).$$

Furthermore,  $\{Q_{k,r} \mid r_{\min}^{(k)} \leq r \leq r_{\max}^{(k)}\}$  forms a basis of  $\mathcal{Q}_k$ .

This result is similar to that proved in Proposition 4 for linear operators: up to a scalar multiple there is only one quadratic autonomous operator with a given order  $r$ , monomial deficiency  $k$ , and deficiency  $m(k, r)$ . In Appendix A we show the explicit form of these operators for  $n = 4, 5$  and  $6$ .

We will prove the theorem by finding a generating function for the operators  $Q_{k,r}$ . This requires some new notation. For a bivariate formal series,

$$p(z, w) = \sum_{i,j \geq 0} p_{ij} z^i w^j,$$

let us define

$$(64) \quad \mathcal{G}\{p(z, w)\} = \sum_{i \geq 0} p_{i,i} z^i$$

to be the series formed from terms where the two variables have equal exponents. We will also adopt the convention that

$$(65) \quad \binom{a}{i} = \begin{cases} \frac{a(a-1) \cdots (a-i+1)}{i!}, & i \geq 0 \\ 0 & i < 0 \end{cases}$$

**Proof**[ Proof of Theorem 2 ] We extend the definition (62) of  $Q_{k,r}$  to all  $0 \leq r \leq n$  by setting

$$(66) \quad Q_{k,r} = \sum_{i=\max(0,k)}^{\min(n,n+k)} (-1)^i \binom{i-k+1}{n-r+1} \binom{n-i}{n-r} u_i u_{n+k-i}.$$

We can now form a generating function for  $Q_{k,r}$  as follows:

(67)

$$\begin{aligned}
Q(z, t) &= \sum_{k=-n}^n \sum_{r=0}^n Q_{k,r} z^{n-r} t^{n-k} \\
&= \sum_{i,j=0}^n \sum_{r=0}^n (-1)^i \binom{n-i}{n-r} \binom{n-j+1}{n-r+1} u_i u_j z^{n-r} t^{2n-i-j} \\
&= \sum_{i,j=0}^n (-1)^i z^{-1} \mathcal{G}\{z(1+z)^{n-i}(1+w)^{n-j+1}\} u_i u_j t^{2n-i-j} \\
(68) \quad &= \sum_{i,j=0}^n \sum_{p,q=0}^n (-1)^i \binom{s}{n-i-p} z^{-1} \mathcal{G}\{z(1+z)^{n-i}(1+w)^{n-j+1}\} I_p I_q \frac{\xi^s}{s!} t^{2n-i-j},
\end{aligned}$$

where  $s = 2n - i - j - p - q$ , and where we used relation (54) to replace the  $u_j$  with the  $I_j$ , the autonomous generators of maximal deficiency defined by (49). Let us extend the definition of  $I_j$  to all  $j \geq 0$  by setting

$$(69) \quad I_j = \sum_{i=0}^j (-1)^i u_{n+i-j} \frac{\xi^i}{i!},$$

and agreeing that  $u_j = 0$  for  $j < 0$ . However, we must note that the newly defined autonomous operators  $I_j$ ,  $j > n$  are no longer maximal deficiency operators. Interchanging the summation order in (68) and re-indexing with

$$i \rightarrow n - p - i, \quad j \rightarrow n - q - j$$

we obtain

$$\begin{aligned}
Q(z, t) &= \sum_{p,q=0}^n \sum_{i=0}^{n-p} \sum_{j=0}^{n-q} (-1)^i \binom{s}{n-p-i} z^{-1} \mathcal{G}\{z(1+z)^{n-i}(1+w)^{n-j+1}\} I_p I_q \frac{\xi^s}{s!} t^{2n-i-j} \\
&= \sum_{p,q=0}^n \sum_{i=0}^{n-p} \sum_{j=0}^{n-q} (-1)^i \binom{s}{i} z^{-1} \mathcal{G}\{z(1+z)^{p+i}(1+w)^{q+j+1}\} I_p I_q \frac{\xi^s}{s!} t^{p+q+s},
\end{aligned}$$

with  $s = i + j$  henceforth. Relation (54) holds for  $j < 0$ , thanks to the extended definition (69) of  $I_j$ ,  $j > n$ . Hence,

$$\begin{aligned}
Q(z, t) &= \sum_{p,q=0}^{2n} \sum_{i=0}^{2n-p-q} \sum_{j=0}^{2n-p-q-i} (-1)^i \binom{s}{i} z^{-1} \mathcal{G}\{z(1+z)^{p+i}(1+w)^{q+j+1}\} I_p I_q \frac{\xi^s}{s!} t^{p+q+s} \\
&= \sum_{p,q=0}^{2n} \sum_{s=0}^{2n-p-q} \sum_{i=0}^s (-1)^{n-p-i} \binom{s}{i} z^{-1} \mathcal{G}\{z(1+z)^{p+i}(1+w)^{q+1+s-i}\} I_p I_q \frac{\xi^s}{s!} t^{p+q+s} \\
(70) \quad &= \sum_{p,q=0}^{2n} \sum_{s=0}^{2n-p-q} (-1)^{n-p} z^{-1} \mathcal{G}\{z(1+z)^p(1+w)^{q+1}(w-z)^s\} I_p I_q \frac{\xi^s}{s!} t^{p+q+s}
\end{aligned}$$

For every positive integer  $s$ , let us introduce the generating function

$$(71) \quad \phi(p, q, s; z) = \mathcal{G}\{(1+z)^p(1+w)^q(w-z)^s\} = \sum_{\rho=0}^{\infty} \phi_{p,q,\rho,s} z^\rho,$$

where

$$(72) \quad \phi_{p,q,\rho,s} = \sum_{i=0}^{\rho} (-1)^{\rho-i} \binom{\rho}{i} \binom{q}{2\rho-s-i} \binom{s}{\rho-i},$$

Thus,

$$\begin{aligned}
(73) \quad & \phi_{p,q,\rho,s} = 0, \quad s > 2\rho, \\
& \phi_{p,q,\rho,2\rho} = (-1)^\rho \binom{2\rho}{\rho}, \\
& \phi_{p,q,\rho,2\rho-1} = (q-p) \binom{2\rho-1}{\rho}, \\
& \phi_{p,q,\rho,2\rho-2} = \left( \binom{q}{2} - \binom{p}{2} \right) \binom{2\rho-2}{\rho} - pq \binom{2\rho-2}{\rho-1}
\end{aligned}$$

Before continuing, let us note the following properties of this function:

$$(74) \quad \phi(p, q, s; z) = (-1)^s \phi(q, p, s; z)$$

$$(75) \quad \phi(p, q, s+1, z) = \phi(p, q+1, s; z) - \phi(p+1, q, s; z)$$

This function is relevant to our proof, because

$$(76) \quad \mathcal{G}\{z(1+z)^p(1+w)^{q+1}(w-z)^s\} = \phi(p+1, q+1, s; z) - \phi(p, q+1, s; z).$$

Hence,

$$(77) \quad Q(z, t) = \sum_{k=-n}^n \sum_{r=0}^n \sum_{s=0}^{2n} Q_{k,r,s} \frac{\xi^s}{s!} z^{n-r} t^{n-k}$$

where

$$(78) \quad Q_{k,r,s} = Q_{k,n-\rho,s} = \sum_{p+q=n-k-s} (-1)^{n-p} (\phi_{p+1,q+1,\rho+1,s} - \phi_{p,q+1,\rho+1,s}) I_p I_q,$$

$$(79) \quad \sum_{r=0}^n Q_{k,r,s} z^{n-r} = \sum_{p+q=n-k-s} (-1)^{n-p} z^{-1} (\phi(p+1, q+1, s; z) - \phi(p, q+1, s; z)) I_p I_q.$$

Let  $m(k, r)$  denote the deficiency of  $Q_{k,r}$ . By Proposition 1,  $m(k, r)$  is equal to  $n$  minus the largest value of  $s$  for which  $Q_{k,r,s} \neq 0$ . By (73) and (78), we know that  $Q_{k,n-\rho,s} = 0$  for  $s > 2\rho + 2$ . For  $s = 2\rho + 2$ , we have

$$\phi_{p+1,q+1,\rho+1,2\rho+2} - \phi_{p,q+1,\rho+1,2\rho+2} = (-1)^{\rho+1} \left( \binom{2\rho+2}{\rho+1} - \binom{2\rho+2}{\rho+1} \right) = 0,$$

and hence,  $Q_{k,n-\rho,s} = 0$  for  $s = 2\rho + 2$ .

The analysis now breaks up into two cases. Suppose that  $n - k$  is even. For  $s = 2\rho + 1$  we again use (73) and (78) to obtain

$$Q_{k,n-\rho,2\rho+1} = \sum_{p+q=n-k-2\rho-1} (-1)^{n-p+\rho+1} \binom{2\rho+1}{\rho} I_p I_q = 0,$$

because of  $p, q$  symmetry. For  $s = 2\rho$ , we have

$$Q_{k,n-\rho,2\rho} = \sum_{p+q=n-k-2\rho} (-1)^{n-p-\rho} \frac{(2\rho+p+q+2)}{2(\rho+1)} \binom{2\rho}{\rho} I_p I_q \neq 0.$$

Therefore, if  $n - k$  is even, we have  $m(k, r) = n - 2\rho = 2r - n$ . This agrees with (63).

Next, suppose that  $n - k$  is odd. Now for  $s = 2\rho + 1$  we have

$$Q_{k,n-\rho,2\rho+1} = \sum_{p+q=n-k-2\rho-1} (-1)^{n-p+\rho+1} \binom{2\rho+1}{\rho} I_p I_q \neq 0,$$

because now  $p$  and  $q$  have the same parity. Therefore, if  $n-k$  is odd, then  $m(k, r) = n - 2\rho - 1 = 2r - n - 1$ . Again, this agrees with (63).  $\square$

## 7. SEPARATION OF VARIABLES IN NON-LINEAR EVOLUTION EQUATIONS

One of the main applications of the results of this paper lies in the method of separation of variables in non-linear evolution equations. Let  $u \in \mathcal{P}_n$  and  $T$  be a non-linear differential operator with deficiency  $m \geq 0$ . This means that  $T\mathcal{P}_n \subset \mathcal{P}_n$ , or more explicitly

$$(80) \quad T \left[ \sum_{i=0}^n p_i x^i \right] = \sum_{i=0}^n f_i(p_1, \dots, p_n) x^i$$

The non-linear evolution equation

$$(81) \quad u_t = T[u] = p(x, u, u_1, \dots, u_n)$$

admits separable solutions of the form

$$(82) \quad u(x, t) = \sum_{i=0}^n \varphi_i(t) x^i,$$

where the functions  $\varphi(t)$  satisfy the following system of first order ordinary differential equations:

$$(83) \quad \dot{\varphi}_i = f_i(\varphi_1, \dots, \varphi_n), \quad i = 0, \dots, n.$$

These results extend immediately to the multivariate case. This technique has been used by King to find new exact multidimensional solutions of non-linear diffusion equations [16, 18] and exact solutions of high order thin film equations [17]. In a more mathematical context, Galaktionov [15] and Svirshchevskii [19] have analyzed non-linear operators that preserve low dimensional spaces spanned by polynomials and trigonometric functions.

**Example 4** Consider the following non-linear evolution equation for  $u = u(t, x)$ :

$$(84) \quad u_t = 7 \left( u u_{xxxx} - \frac{5}{2} u_x u_{xxx} + \frac{45}{28} u_{xx}^2 \right)$$

This equation has solutions in  $\mathcal{P}_8$  of the form

$$(85) \quad u(x, t) = \sum_{i=0}^8 \varphi_i(t) x^i,$$

where the functions  $\varphi_i(t)$  satisfy the following first order system:

$$\begin{aligned} \dot{\varphi}_0 &= 45 \varphi_2^2 - 105 \varphi_1 \varphi_3 + 168 \varphi_0 \varphi_4 \\ \dot{\varphi}_1 &= 60 \varphi_2 \varphi_3 - 252 \varphi_1 \varphi_4 + 840 \varphi_0 \varphi_5 \\ \dot{\varphi}_2 &= 90 \varphi_3^3 - 132 \varphi_2 \varphi_4 - 210 \varphi_1 \varphi_5 + 2520 \varphi_0 \varphi_6 \\ \dot{\varphi}_3 &= 108 \varphi_3 \varphi_4 - 360 \varphi_2 \varphi_5 + 420 \varphi_1 \varphi_6 + 5880 \varphi_0 \varphi_7 \\ \dot{\varphi}_4 &= 108 \varphi_4^2 - 135 \varphi_3 \varphi_5 - 330 \varphi_2 \varphi_6 + 2205 \varphi_1 \varphi_7 + 11760 \varphi_0 \varphi_8 \\ \dot{\varphi}_5 &= 108 \varphi_4 \varphi_5 - 360 \varphi_3 \varphi_6 + 420 \varphi_2 \varphi_7 + 5880 \varphi_1 \varphi_8 \\ \dot{\varphi}_6 &= 90 \varphi_5^3 - 132 \varphi_4 \varphi_6 - 210 \varphi_3 \varphi_7 + 2520 \varphi_2 \varphi_8 \\ \dot{\varphi}_7 &= 60 \varphi_5 \varphi_6 - 252 \varphi_4 \varphi_7 + 840 \varphi_3 \varphi_8 \\ \dot{\varphi}_8 &= 45 \varphi_6^2 - 105 \varphi_5 \varphi_7 + 168 \varphi_4 \varphi_8 \end{aligned}$$

## APPENDIX A

In this Appendix we list the specialized autonomous operator basis  $Q_{k,r}$  described in Theorem 1. Recall that each  $Q_{k,r}$  has monomial deficiency  $k$ , order  $r$ , and deficiency  $m = m(k, r)$ . The following tables for  $n = 4, 5, 6$  display the non-linear operators with  $\{k, r, m\}$  in the ranges

$$-n \leq k \leq n, \quad \left\lceil \frac{n+k}{2} \right\rceil \leq r \leq \min(n, n+k), \quad m = k + 2(r - r_{\min}).$$

Recall that an operator with deficiency  $m$  maps  $\mathcal{P}_n$  to  $\mathcal{P}_{n-m}$ , but not  $\mathcal{P}_{n-m-1}$ . Hence all operators with  $m \geq 0$  map  $\mathcal{P}_n$  to  $\mathcal{P}_n$  and can be used to construct an evolution equation solvable by non-linear separation of variables.

TABLE 1. Quadratic autonomous operators acting on  $\mathcal{P}_4$ 

$k$	$Q_{k,r}$	$m$
-4	$u_0^2$	-4
-3	$u_0 u_1$	-3
-2	$u_1^2 \quad 3u_1^2 - 4u_0 u_2$	-2 0
-1	$u_1 u_2 \quad u_1 u_2 - 2u_0 u_3$	-1 1
0	$u_2^2 \quad 2u_2^2 - 3u_1 u_3 \quad u_2^2 - 2u_1 u_3 + 2u_0 u_4$	0 2 4
1	$u_2 u_3 \quad u_2 u_3 - 3u_1 u_4$	1 3
2	$u_3^2 \quad u_3^2 - 2u_2 u_4$	2 4
3	$u_3 u_4$	3
4	$u_4^2$	4

TABLE 2. Quadratic autonomous operators acting on  $\mathcal{P}_5$ 

$k$	$Q_{k,r}$	$m$
-5	$u_0^2$	-5
-4	$u_0 u_1$	-4
-3	$u_1^2 \quad 4u_1^2 - 5u_0 u_2$	-3 -1
-2	$u_1 u_2 \quad 3u_1 u_2 - 5u_0 u_3$	-2 0
-1	$u_2^2 \quad 3u_2^2 - 4u_1 u_3 \quad 9u_2^2 - 16u_1 u_3 + 10u_0 u_4$	-1 1 3
0	$u_2 u_3 \quad u_2 u_3 - 2u_1 u_4 \quad u_2 u_3 - 3u_1 u_4 + 5u_0 u_5$	0 2 4
1	$u_3^2 \quad 2u_3^2 - 3u_2 u_4 \quad u_3^2 - 2u_2 u_4 + 2u_1 u_5$	1 3 5
2	$u_3 u_4 \quad u_3 u_4 - 3u_2 u_5$	2 4
3	$u_4^2 \quad u_4^2 - 2u_3 u_5$	3 5
4	$u_4 u_5$	4
5	$u_5^2$	5

TABLE 3. Quadratic autonomous operators acting on  $\mathcal{P}_6$ 

$k$	$Q_{k,r}$	$m$
-6	$u_0^2$	-6
-5	$u_0 u_1$	-5
-4	$u_1^2 - 5u_0 u_2$	-4 -2
-3	$u_1 u_2 - 2u_0 u_3$	-3 -1
-2	$u_2^2 - 4u_1 u_3 - 6u_0 u_4$	-2 0 2
-1	$u_2 u_3 - 3u_1 u_4 - 5u_0 u_5$	-1 1 3
0	$u_3^2 - 3u_2 u_4 - 4u_1 u_5 - 9u_0 u_6$	0 2 4 6
1	$u_3 u_4 - 2u_2 u_5 - 3u_1 u_6$	1 3 5
2	$u_4^2 - 2u_3 u_5 - 2u_2 u_6$	2 4 6
3	$u_4 u_5 - 3u_3 u_6$	3 5
4	$u_5^2 - 2u_4 u_6$	4 6
5	$u_5 u_6$	5
6	$u_6^2$	6

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